

# THE TOPOLOGY OF THE MOMENT-ANGLE MANIFOLDS —ON A CONJECTURE OF S.GITLER AND S.LÓPEZ

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**ABSTRACT.** Let  $P$  be a simple polytope of dimension  $n$  with  $m$  facets and  $P_v$  be a polytope obtained from  $P$  by cutting off one vertex  $v$ . Let  $Z = Z(P)$  and  $Z_v = Z(P_v)$  be the corresponding moment-angle manifolds. In [GL] S.Gitler and S.López conjectured that:  $Z_v$  is diffeomorphic to  $\partial[(Z - \text{int}(D^{n+m})) \times D^2] \#_{j=1}^{m-n} \#_{j=1}^{m-n} (S^{j+2} \times S^{m+n-j-1})$ , and they have proved the conjecture in the case  $m < 3n$ . In this paper we prove the conjecture in general case.

## 1. INTRODUCTION

The moment-angle manifold  $Z$  comes from two different ways:

- (1) The transverse intersections in  $\mathbb{C}^n$  of real quadrics of the form  $\sum_{i=1}^n a_i |z_i|^2 = 0$  with the unit euclidean sphere of  $\mathbb{C}^n$ .
- (2) An abstract construction from a simple polytope  $P^n$  with  $m$ -facets (or a complex  $K$ ).

The study of the first one led to the discovery of a new special class of compact non-kähler complex manifolds in the work of Lopez, Verjovsky and Meersseman ([LV],[Me],[MV]), now known as the LV-M manifolds, which helps us understand the topology of non-kähler complex manifolds.

The study of the second one is related to the quasitoric manifolds in the following way: for every quasitoric manifold  $\pi : M^{2n} \rightarrow P^n$ , there is a principal  $T^{m-n}$ -bundle  $Z \rightarrow M^{2n}$  whose composite map with  $\pi$  makes  $Z$  a  $T^m$ -manifold with orbit space  $P^n$ . The topology of the manifolds  $Z$  provides an effective tool for understanding inter-relations between algebraic and combinatorial aspects such as the Stanley-Reisner rings, the subspace arrangements and the cubical complexes etc.(see [BP]).

Following [BP], let  $P$  be an  $n$ -dimensional simple polytope with  $m$  facets,  $K_P$  be the dual of the boundary of  $P$ . Obviously,  $K_P$  is a simplicial complex. Let  $[m] = \{0, 1, \dots, m-1\}$  represent the  $m$  vertices of the simplicial complex,  $\sigma$  be a simplex in the complex  $K_P$ . Define

$$(D^2)_\sigma \times T_{\widehat{\sigma}} = \{(z_1, z_2, \dots, z_m) \in (D^2)^m : |z_j| = 1 \text{ for } j \notin \sigma\}.$$

and define the moment-angle complex  $Z(P)$  corresponding to  $P$  as

$$Z(P) = \bigcup_{\sigma \in K_P} (D^2)_\sigma \times T_{\widehat{\sigma}} \subset (D^2)^m.$$

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A fundamental problem is to study the topology of  $Z(P)$  corresponding to some  $P$ . One way to do this is to consider the change of the moment-angle complex  $Z$  after we do some 'surgery' on the polytope  $P$ . These 'surgeries' include bistellar moves, cutting off faces, etc. (see [BP]). Obviously if we can obtain some information about the change of the topology of moment-angle complex after the 'surgery', we will get a better understanding of the moment-angle manifolds. However, cutting off vertices is the simplest 'surgery' and seems to be the only 'surgery' can be totally understood according to the conjecture of Gitler and López.

Let  $P$  be a simple polytope of dimension  $n$  with  $m$  facets, which is the convex hull of finitely many vertices in  $\mathbb{R}^n$ . For any vertex  $v$ , we can find a hyperplane  $H(x) = \sum_{i=1}^n a_i x_i = b$  satisfying that  $H(v) > b$  and  $H(\widehat{v}) < b$  for any other vertex  $\widehat{v}$ . The set  $P \cap \{x | H(x) \leq b\}$  is a new simple polytope  $P_v$ , which is called to be obtained from  $P$  by cutting off the vertex  $v$ . Let  $Z = Z(P)$  and  $Z_v = Z(P_v)$  be the corresponding moment-angle manifolds. From [BP] (6.9) we know that

$$Z_v = \partial \left[ (Z - T_{\sigma}^{m-n} \times \text{int}(D_{\sigma}^{2n})) \times D^2 \right],$$

where  $\sigma$  is the simplex dual to the vertex  $v$ . S. Gitler and S. López conjectured that  $Z_v$  is diffeomorphic to

$$\partial[(Z - \text{int}(D^{n+m})) \times D^2] \#_{j=1}^{m-n} \binom{m-n}{j} (S^{j+2} \times S^{m+n-j-1})$$

and they proved the conjecture in the case of  $m < 3n$  (see [GL]).

In the case  $m < 3n$  ([GL]), S. Gitler and S. López firstly proved that  $T_{\sigma}^{m-n}$  is isotopic to a torus  $T^{m-n}$  inside an open disk in  $Z$ . Then from the isotopy extension theorem (see [Ko]), one can prove that

$$Z - T_{\sigma}^{m-n} \times \text{int}(D_{\sigma}^{2n}) \cong Z - T^{m-n} \times \text{int}(D^{2n}) \cong (Z - \text{int}(D^{n+m})) \cup (D^{n+m} - T^{m-n} \times \text{int}(D^{2n}))$$

and

$$Z_v = \partial[(Z - \text{int}(D^{n+m})) \times D^2] \# \partial[(S^{m+n} - T^{m-n} \times \text{int}(D^{2n})) \times D^2].$$

Secondly they considered the manifold  $\partial[(S^{m+n} - T^{m-n} \times \text{int}(D^{2n})) \times D^2]$ . They constructed spheres represent the Alexander dual homology of  $T^{m-n}$  in  $S^{m+n}$ . According to a corollary of  $h$ -cobordism theorem, they proved that

$$\partial[(S^{m+n} - T^{m-n} \times \text{int}(D^{2n})) \times D^2] = \#_{j=1}^{m-n} \binom{m-n}{j} (S^{j+2} \times S^{m+n-j-1}).$$

So the conjecture is true in this case.

In this paper, we firstly construct an isotopy of  $T_{\sigma}^{m-n}$  in  $Z$  to move it to the regular embedding  $T^{m-n} \subseteq D^{m-n+1} \subseteq D^{m+n} \subseteq Z$ , thus  $Z_v$  is diffeomorphic to

$$\partial[(Z - \text{int}(D^{n+m})) \times D^2] \# \partial[(S^{m+n} - T^{m-n} \times \text{int}(D^{2n})) \times D^2]$$

in general. Secondly by Lemma 2.13 in [Mc], we can prove that  $\partial[(S^{m+n} - T^{m-n} \times \text{int}(D^{2n})) \times D^2]$  is diffeomorphic to  $\#_{j=1}^{m-n} \binom{m-n}{j} (S^{j+2} \times S^{m+n-j-1})$ .

The main result of the paper is:

**Theorem 1.1.** *let  $P$  be a simple polytope of dimension  $n$  with  $m$  facets and  $P_v$  be a polytope obtained from  $P$  by cutting off one vertex  $v$ . Let  $Z = Z(P)$  and  $Z_v = Z(P_v)$  be the corresponding moment-angle manifolds, then  $Z_v$  is diffeomorphic to*

$$\partial[(Z - \text{int}(D^{n+m})) \times D^2] \#_{j=1}^{m-n} \binom{m-n}{j} (S^{j+2} \times S^{m+n-j-1}).$$

The manifold  $\partial[(Z - \text{int}(D^{m+n})) \times D^2]$  is diffeomorphic to  $(Z \times S^1 - \text{int}(D^{m+n}) \times S^1) \cup S^{m+n-1} \times D^2$ , which can be obtained by doing a  $(m+n, 1)$ -type surgery on the manifold  $Z \times S^1$  (see [M]).

**Proposition 1.1.** *Let  $[Z]$  and  $[S^1]$  be the fundamental classes of  $Z$  and  $S^1$  respectively. Then the cohomology of  $\partial[(Z - \text{int}(D^{m+n})) \times D^2]$  is isomorphic to*

$$H^*(\partial[(Z - \text{int}(D^{m+n})) \times D^2]) \cong H^*(Z) \otimes H^*(S^1) / \{1 \otimes [S^1], [Z] \otimes 1\}$$

as a ring.

## 2. CONSTRUCT THE ISOTOPY OF $T_{\widehat{\sigma}}^{m-n} \times 0$ IN $Z$

After cutting off a vertex  $v$  on the simple polytope  $P$ , we obtain a new simple polytope  $P_v$ . Let  $K_P$  and  $K_{P_v}$  be the duals of the boundary of  $P$  and  $P_v$ ,  $\sigma$  be the maximal simplex in  $K_P$  dual to the vertex  $v$  of the simple polytope  $P$ . Then we have  $K_{P_v} = K_P \#_{\sigma} \partial \Delta^n$  ( $\Delta^n$  is the standard  $n$ -dimensional simplex, the choice of a maximal simplex in  $\partial \Delta^n$  is irrelevant). By the definition, the moment-angle complex corresponding to  $P$  (or  $K_P$ ) is:

$$Z = \bigcup_{\sigma \in K_P} (D^2)_{\sigma} \times T_{\widehat{\sigma}} \subset (D^2)^m.$$

Then we can express the moment-angle complex corresponding to  $P_v$  (or  $K_{P_v}$ ) as follows (see 6.4 in [Mc]):

$$\begin{aligned} Z_v &= (Z \times S^1 - T_{\widehat{\sigma}}^{m-n} \times \text{int}(D_{\sigma}^{2n}) \times S^1) \cup_{T_{\widehat{\sigma}}^{m-n} \times S_{\sigma}^{2n-1} \times S^1} T_{\widehat{\sigma}}^{m-n} \times S_{\sigma}^{2n-1} \times D^2 \\ &\simeq \partial[(Z - T_{\widehat{\sigma}}^{m-n} \times \text{int}(D_{\sigma}^{2n})) \times D^2] \end{aligned} \quad (1)$$

Without loss of generality, assume  $\widehat{\sigma}$  correspond to the vertices  $\{1, \dots, m-n\}$ ,  $*$  be a point of  $S_{m-n+1}^1 \times \dots \times S_{m-1}^1$ ,  $y$  be a point of  $S_0^1$ . In this section, we construct an isotopy inductively to move the torus  $\{y\} \times S_1^1 \times \dots \times S_{m-n}^1 \times \{*\}$  in  $Z$  to the regular embedding  $T^{m-n} \subseteq D^{m-n+1} \subseteq D^{m+n} \subseteq Z$ .

**Remark 2.1.** *We construct the regular embedding of  $T^k$  into  $\mathbb{R}^{k+1}$  as follows:  $S^1 \subseteq D^2 \subseteq \mathbb{R}^2$ , assume that we have constructed the embedding of  $T^{i-1}$  into  $D^i \subseteq \mathbb{R}^i$ . Represent  $(i+1)$ -sphere as  $S^{i+1} = D^i \times S^1 \cup S^{i-1} \times D^2$ . By the assumption, the torus  $T^i = T^{i-1} \times S^1$  can be embedded into  $D^i \times S^1$  and therefore into  $S^{i+1}$ . Since  $T^i$  is compact and  $S^{i+1}$  is the one-point compactification of  $\mathbb{R}^{i+1}$ , we have  $T^i \subseteq \mathbb{R}^{i+1}$ . Inductively, we can construct the regular embedding of  $T^k$  into  $\mathbb{R}^{k+1}$  (or  $D^{k+1}$ ). The regular embedding of  $T^k$  into  $\mathbb{R}^n$  is  $T^k \subseteq \mathbb{R}^{k+1} \times \{0\} \subseteq \mathbb{R}^{k+1} \times \mathbb{R}^{n-k-1}$ , where  $T^k \subseteq \mathbb{R}^{k+1} \times \{0\}$  is the regular embedding of  $T^k$  into  $\mathbb{R}^{k+1}$ .*

In terms of coordinates, we can express the regular embedding torus  $T^k$  in  $\mathbb{R}^{k+1}$  inductively as:

$$T^k = \left\{ \begin{pmatrix} \sin \alpha_1 \cdot (1 + \frac{1}{2} \sin \alpha_2 \cdot (1 + \frac{1}{2} \sin \alpha_3 \cdot (\cdots (1 + \frac{1}{2} \sin \alpha_k) \cdots))) \\ \cos \alpha_1 \cdot (1 + \frac{1}{2} \sin \alpha_2 \cdot (1 + \frac{1}{2} \sin \alpha_3 \cdot (\cdots (1 + \frac{1}{2} \sin \alpha_k) \cdots))) \\ \frac{1}{2} \cos \alpha_2 \cdot (1 + \frac{1}{2} \sin \alpha_3 \cdot (\cdots (1 + \frac{1}{2} \sin \alpha_k) \cdots)) \\ \vdots \\ \frac{1}{2^{k-2}} \cos \alpha_{k-1} \cdot (1 + \frac{1}{2} \sin \alpha_k) \\ \frac{1}{2^{k-1}} \cos \alpha_k \end{pmatrix} \mid 0 \leq \alpha_i < 2\pi \right\} \quad (2)$$

We shall call this the standard torus  $T^k \subseteq \mathbb{R}^{k+1}$ . Consider the isotopy  $F : T^k \times I \longrightarrow D^{k+2}$  defined by:

$$F(\alpha, t) = \begin{pmatrix} \sin \alpha_1 \cdot (1 + \frac{1}{2} \sin \alpha_2 \cdot (1 + \frac{1}{2} \sin \alpha_3 \cdot (\cdots (1 + \frac{1}{2} t \sin \alpha_k) \cdots))) \\ \cos \alpha_1 \cdot (1 + \frac{1}{2} \sin \alpha_2 \cdot (1 + \frac{1}{2} \sin \alpha_3 \cdot (\cdots (1 + \frac{1}{2} t \sin \alpha_k) \cdots))) \\ \frac{1}{2} \cos \alpha_2 \cdot (1 + \frac{1}{2} \sin \alpha_3 \cdot (\cdots (1 + \frac{1}{2} t \sin \alpha_k) \cdots)) \\ \vdots \\ \frac{1}{2^{k-2}} \cos \alpha_{k-1} \cdot (1 + \frac{1}{2} t \sin \alpha_k) \\ \frac{1}{2^{k-1}} \cos \alpha_k \\ \frac{1}{2^{k-1}} [(1-t) \sin \alpha_k + t |\sin \alpha_k|] \end{pmatrix}. \quad (3)$$

An examination of this isotopy proves the following :

**Lemma 2.1.** *Let  $T^k$  be the standard torus in  $D^{k+1} = D^k \times D^1 \subseteq D^k \times S^1 \subseteq D^k \times D^2$ . Then we may write  $D^{k+2}$  as  $D^k \times D^2$  so that  $T^k$  is isotopic to  $T^{k-1} \times (\partial D^2)$ , where  $T^{k-1}$  is a standard torus in  $D^k$ .*

*Proof.* While  $t = 0$ , the last coordinate  $|\sin \alpha_k|$  in (3) is omitted. □

Now we can construct an isotopy of torus  $T_{\sigma}^{m-n} \times \{0\}$  in  $Z$ :

Obviously, the torus  $\{0\} \times T_{\sigma}^{m-n} \times \{0\}$  in  $Z$  is isotopic to  $\{y\} \times T_{\sigma}^{m-n} \times \{*\}$ .

As  $n \geq 1$ , the torus

$$S_0^1 \times T_{\sigma}^{m-n} \times \{*\} \subseteq Z = \bigcup_{\sigma \in K_P} (D^2)_{\sigma} \times T_{\sigma}^{m-n} \subset (D^2)^m.$$

Assume the facet  $\sigma_1$  contain vertex 1, so

$$S_0^1 \times T_{\sigma}^{m-n} \times \{*\} \subseteq S_0^1 \times D_1^2 \times S_2^1 \times \cdots \times S_{m-n}^1 \times \{*\} \subseteq (D^2)_{\sigma_1} \times T_{\sigma_1} \subseteq Z.$$

For  $\forall x \in S_2^1 \times \cdots \times S_{m-n}^1$ , we construct an isotopy of  $\{y\} \times S_1^1 \times \{x\} \times \{*\}$  in  $S_0^1 \times D_1^2 \times \{x\} \times \{*\}$  as follows: Let  $D_0^1$  be a closed interval of  $S_0^1$  with original point  $y = 0$ . The coordinate of a point of  $\{y\} \times S_1^1$  in  $D_0^1 \times D_1^2 \subset S_0^1 \times D_1^2$  is  $(0, \cos \alpha, \sin \alpha)$ . Define

$$F_1 : S_1^1 \times I \longrightarrow D_0^1 \times D_1^2$$

$$F_1(\cos \alpha, \sin \alpha, t) = (t \sin \alpha, \cos \alpha, (1-t) \sin \alpha + t |\sin \alpha|)$$

when  $t = 1$ ,  $F_1(\cos \alpha, \sin \alpha, 1) = (\sin \alpha, \cos \alpha, |\sin \alpha|)$ . Obviously,  $(\sin \alpha, \cos \alpha, |\sin \alpha|)$  is the regular embedding  $S^1 = \partial D^2 \subseteq D^2 \subseteq S_0^1 \times S_1^1$ , where  $D^2$  is a smooth embedded disk in  $S_0^1 \times S_1^1$ .

In this way, for  $\forall x \in S_2^1 \times \cdots \times S_{m-n}^1$ , we move  $\{y\} \times S_1^1 \times \{x\} \times \{*\}$  to the regular embedding

$$S^1 \subset D^2 \subset S_0^1 \times S_1^1 \times \{x\} \times \{*\}.$$

This gives an isotopy to move  $\{y\} \times S_1^1 \times S_2^1 \times \dots \times S_{m-n}^1 \times \{*\}$  into

$$D^2 \times (S_2^1 \times \dots \times S_{m-n}^1) \times \{*\} \subset S_0^1 \times S_1^1 \times (S_2^1 \times \dots \times S_{m-n}^1) \times \{*\},$$

where  $D^2 \subset S_0^1 \times S_1^1$ .

Inductively suppose we have constructed an isotopy of  $\{y\} \times S_1^1 \times \dots \times S_p^1 \times \{x\} \times \{*\}$  to move it to the regular embedding

$$T^p \subseteq D^{p+1} \subseteq S_0^1 \times S_1^1 \times \dots \times S_p^1 \times \{x\} \times \{*\}$$

where  $x$  is a point of  $S_{p+1}^1 \times \dots \times S_{m-n}^1$  and the coordinate of the points of  $T^p \subset D^{p+1}$  is expressed as (2). Assume the facet  $\sigma_{p+1}$  contains vertex  $p+1$ , so

$$S_0^1 \times S_1^1 \times \dots \times S_p^1 \times D_{p+1}^2 \times \{\pi(x)\} \times \{*\} \subset (D^2)_{\sigma_{p+1}} \times T_{\widehat{\sigma_{p+1}}} \subseteq Z,$$

where  $\pi$  is the projection

$$\pi : S_{p+1}^1 \times S_{p+2}^1 \times \dots \times S_{m-n}^1 \rightarrow S_{p+2}^1 \times \dots \times S_{m-n}^1.$$

Using Lemma 2.1 above, we can construct an isotopy to move the torus

$$T^p (\subseteq D^{p+1}) \times S_{p+1}^1 (\subseteq D_{p+1}^2) \times \{\pi(x)\} \times \{*\}$$

to the regular embedding

$$T^{p+1} \subseteq D^{p+2} \subseteq S_0^1 \times S_1^1 \times \dots \times S_{p+1}^1 \times \{\pi(x)\} \times \{*\}.$$

In this way, we can construct an isotopy  $F_t$  of  $T_{\widehat{\sigma}}^{m-n} \times \{0\} \subseteq Z$  to move it to the regular embedding  $T^{m-n} \subseteq D^{m-n+1} \subseteq D^{m+n} \subseteq Z$ . According to the isotopy extension theorem, there exists an isotopy  $G_t$  of  $Z$  satisfying  $G_t|_{T_{\widehat{\sigma}}^{m-n} \times \{0\}} = F_t$ . So the proper tubular neighborhood  $T_{\widehat{\sigma}}^{m-n} \times \text{int}(D^{2n})$  of  $T_{\widehat{\sigma}}^{m-n} \times \{0\}$  in  $Z$  is isotopic to a proper tubular neighborhood  $N(T^{m-n})$  of  $T^{m-n}$  ( $T^{m-n}$  is a regular embedding  $T^{m-n} \subseteq D^{m+n} \subseteq Z$ ). By Theorem 3.5 in [Ko],  $N(T^{m-n})$  can be chosen to be  $T^{m-n} \times \text{int}(D^{m+n}) \subseteq D^{m+n} \subseteq Z$ . So

$$Z - T_{\widehat{\sigma}}^{m-n} \times \text{int}(D^{2n}) \simeq Z \# (S^{m+n} - T^{m-n} \times \text{int}(D^{2n})),$$

$$Z_v \simeq \partial[(Z \# (S^{m+n} - T^{m-n} \times \text{int}(D^{2n}))) \times D^2].$$

Recall Lemma 2 in [GL]:

**Lemma 2.2.** (Lemma 2 [GL]) *Let  $M, N$  be connected  $n$ -manifolds, if  $M$  is closed but  $N$  has non-empty boundary, then  $\partial[(M \# N) \times D^2]$  is diffeomorphic to  $\partial[(M - \text{int}(D^n)) \times D^2] \# \partial(N \times D^2)$ .*

According to the lemma,  $\partial[(Z \# (S^{m+n} - T^{m-n} \times \text{int}(D^{2n}))) \times D^2]$  is diffeomorphic to

$$\partial[(Z - \text{int}(D^{m+n})) \times D^2] \# \partial[(S^{m+n} - T^{m-n} \times \text{int}(D^{2n})) \times D^2],$$

where torus  $T^{m-n} \subseteq D^{m-n+1} \subseteq D^{m+n} \subseteq S^{m+n}$  is the regular embedding.

### 3. THE MANIFOLD $\partial[(S^{m+n} - T^{m-n} \times \text{int}(D^{2n})) \times D^2]$

In order to prove the conjecture, it is sufficient to show that the manifold

$$\partial[(S^{m+n} - T^{m-n} \times \text{int}(D^{2n})) \times D^2]$$

is diffeomorphic to  $\#_{j=1}^{m-n} \binom{m-n}{j} (S^{j+2} \times S^{m+n-j-1})$ .

Let  $M$  and  $N$  be  $m$ -manifolds with boundary, and  $X$  be a closed  $n$ -manifold. Let  $X \times D^{m-n-1}$  be embedded in  $\partial M$  and  $\partial N$ . Then we can form  $(M, F) \#_X (N, G) = M \cup_f N$  (denoted as  $M \#_X N$ ), where  $F, G$  are framings  $(X, x_0) \rightarrow (SO(m-n-1), id)$ ,

$$f : X \times D^{m-n-1} (\subseteq \partial M) \rightarrow X \times D^{m-n-1} (\subseteq \partial N)$$

$$f(x, y) = (x, G(x)F^{-1}(x)(y))$$

It is not difficult to see that

$$\partial M \#_X \partial N = (\partial M - X \times \text{int}(D^{m-n-1})) \cup_{\partial f} (\partial N - X \times \text{int}(D^{m-n-1})) \simeq \partial(M \#_X N)$$

So  $\partial[(S^{m+n} - T^{m-n} \times \text{int}(D^{2n})) \times D^2]$  ( $T^{m-n} \subseteq S^{m+n}$  is the regular embedding) is diffeomorphic to

$$\begin{aligned} & (S^{m+n} \times S^1 - T^{m-n} \times \text{int}(D^{2n}) \times S^1) \cup T^{m-n} \times S^{2n-1} \times D^2 \\ \simeq & (S^{m+n} \times S^1 - T^{m-n} \times \text{int}(D^{2n}) \times S^1) \cup (T^{m-n} \times S^{2n+1} - T^{m-n} \times \text{int}(D^{2n}) \times S^1) \\ \simeq & S^{2n+1} \times T^{m-n} \#_{T^{m-n+1}} S^1 \times S^{m+n} \\ \simeq & \partial(D^{2n+2} \times T^{m-n} \#_{T^{m-n+1}} S^1 \times D^{m+n+1}), \end{aligned} \quad (4)$$

where  $T^{m-n+1} \times D^{2n}$  is embedded in  $\partial(S^1 \times D^{m+n+1})$  through

$$S^1 \times (T^{m-n} \times D^4) \times D^{2n-4} \subseteq S^1 \times S^{m-n+4} \times D^{2n-4} \subseteq S^1 \times S^{m+n} = \partial(S^1 \times D^{m+n+1}),$$

and  $T^{m-n+1} \times D^{2n}$  is embedded in  $\partial(D^{2n+2} \times T^{m-n})$  through

$$T^{m-n} \times (S^1 \times D^4) \times D^{2n-4} \subseteq T^{m-n} \times S^5 \times D^{2n-4} \subseteq T^{m-n} \times S^{2n+1} = \partial(D^{2n+2} \times T^{m-n}).$$

Thus,

$$\begin{aligned} & \partial(D^{2n+2} \times T^{m-n} \#_{T^{m-n+1}} S^1 \times D^{m+n+1}) \\ \simeq & \partial(D^6 \times T^{m-n} \times D^{2n-4} \cup_{T^{m-n+1} \times D^4 \times D^{2n-4}} S^1 \times D^{m-n+5} \times D^{2n-4}) \\ \simeq & \partial[(D^6 \times T^{m-n} \#_{T^{m-n+1}} S^1 \times D^{m-n+5}) \times D^{2n-4}] \end{aligned} \quad (5)$$

Recall Lemma 2.13 in [Mc]:

**Lemma 3.1.** (Lemma 2.13 [Mc]) Let  $T^{n-4} = S^1 \times T^{n-5}$ . Suppose we have product embeddings  $S^1 \times (T^{n-5} \times D^4) \rightarrow S^1 \times D^n$  and  $(S^1 \times D^4) \times T^{n-5} \rightarrow D^6 \times T^{n-5}$  (where  $T^{n-5} \rightarrow D^n$  is the regular embedding). With these embeddings

$$D^6 \times T^{n-5} \#_{T^{n-4}} S^1 \times D^n \simeq \#_{j=1}^{n-5} \binom{n-5}{j} S^{j+2} \times D^{n-j-1}$$

By Lemma 3.1,

$$\begin{aligned}
 & \partial[(D^6 \times T^{m-n} \#_{T^{m-n+1}} S^1 \times D^{m-n+5}) \times D^{2n-4}] \\
 & \simeq \partial\left[\#_{j=1}^{m-n} \binom{m-n}{j} S^{j+2} \times D^{m+n-j}\right] \\
 & \simeq \#_{j=1}^{m-n} \binom{m-n}{j} S^{j+2} \times S^{m+n-j-1}.
 \end{aligned} \tag{6}$$

Thus,  $\partial[(S^{m+n} - T^{m-n} \times \text{int}(D^{2n})) \times D^2] \simeq \#_{j=1}^{m-n} \binom{m-n}{j} S^{j+2} \times S^{m+n-j-1}$ .

$$\begin{aligned}
 Z_v & \simeq \partial[(Z - \text{int}(D^{m+n})) \times D^2] \# \partial[(S^{m+n} - T^{m-n} \times \text{int}(D^{2n})) \times D^2] \\
 & \simeq \partial[(Z - \text{int}(D^{m+n})) \times D^2] \# \#_{j=1}^{m-n} \binom{m-n}{j} (S^{j+2} \times S^{m+n-j-1})
 \end{aligned} \tag{7}$$

Until now, we have proved the conjecture. In a subsequent paper we will discuss the more general problems: the topology of the moment-angle manifold corresponding to the connected sums, bistellar moves and cutting off high dimensional faces.

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